Integrable multi atom matter-radiation models without rotating wave approximation

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Interacting matter-radiation models close to physical systems are proposed, which without rotating wave approximation and with matter-matter interactions are Bethe ansatz solvable. This integrable system is constructed from the elliptic Gaudin model at high spin limit, where radiative excitation can be included perturbatively.

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In dealing with interacting matter-radiation systems the counter rotating wave (CRW) terms, which inevitably appear in general situation, are usually neglected invoking the rotating wave (RW) approximation, without which physical models generally become unsolvable. However the RW approximation breaks down away from the resonance condition at $\omega_a \approx \omega_f$ as well as for high intensity fields. Consequently, fast oscillations (with frequency $\omega_a + \omega_f$) can no longer be neglected compared to the slow ones (with frequency $|\omega_a - \omega_f|$) and one is forced to consider the general case with CRW terms. Moreover, since the situation is generic, this problem can arise in diverse models like those in quantum optics, in cavity QED both in microwave and optical domain [1, 2], in trapped ions irradiated by laser beams [3] as well as in transport through quantum dots coupled to boson model [4] and in quantum information transfer protocol with a superconducting circuit [5].

Under RW approximation one can obtain exactly solvable models like Jayenes-Cummings (JC) [6], Buck-Sukumar [7] model and their multi-atom generalizations [8–10] as well as q-deformed matter-radiation models, inducing anisotropic together with higher nonlinearity [10, 11]. However when this approximation is avoided, CRW terms appear having the form like $H_{crw} = \beta(b^\dagger \sigma^+ + b \sigma^-)$, in the simplest case and generally spoil the solvablity of the system. Models with CRW terms in various forms were investigated earlier [3, 12, 13], though such exactly solvable multi-atom models which are close to physical systems are not known in the literature, except perhaps a recent proposal [14] involving some unphysical terms.

We propose here integrable multi-atom JC type matter-radiation models which include CRW terms and allow exact Bethe ansatz solutions for the vanishing radiation frequency, when the RW approximation is naturally not applicable. Field excitation term can be considered over the integrable model, perturbatively taking ω_f to be small, while the interatomic interactions through spin-spin coupling can be included in an exact way. We derive our model from the elliptic Gaudin model [15], through spin-$\frac{1}{2}$ representation for the atoms and the bosonic realization under high spin limit for the single mode radiation field. For following the logic of the construction, we recall that all known exactly solvable multi-atom matter-radiation models *with* RW approximation are derived from integrable xxx, xxz spin models [8–10] or from their corresponding Gaudin limits [16, 17]. Note that at higher spins the underlying algebras associated with these models are either $su(2), su(1, 1)$ or their quantum deformations, both allowing the crucial bosonic realization. For possible construction of integrable models *without* RW approximation, one may therefore expect to repeat the same construction starting from a more general integrable inhomogeneous xyz model with higher spin representation [18, 19]. The representative Lax operator of this model depends, apart from the spectral parameter λ and the anisotropy parameter α, on the elliptic modulus k, inhomogeneity parameters z_n as well as on the spin s representation, through its dependence on coefficients $W_p(\lambda - z_n; \alpha, k)$, $p = 0, 1, 2, 3$, expressed through elliptic functions and operators $S^p(s, \alpha, k)$ satisfying the Sklyanin algebra [20]. In fact, all known integrable matter-radiation models mentioned above can be obtained from this general structure at different limits of the parameters involved. For example, at $k = 0$, one recovers the trigonometric xxz case, when generators of the Sklyanin algebra reduce to those of the quantum algebra: $S^a(s, \alpha), a = 1, 2, 3$. Under a further limit of $\alpha \to 0$ one obtains the trigonometric Gaudin model with the operators reducing to $S^a(s)$, satisfying the standard Lie algebra. If however at the same time a limit $\lambda \to 0$ is imposed on the spectral parameter (together with z_n), we recover the xxx case as well as the corresponding rational Gaudin model.

We however keep the elliptic modulus k nontrivial throughout our construction and thereby follow a rout different from the earlier ones. Observe that, unlike the above cases the Sklyanin algebra does not have any known bosonic realization, though fortunately at

$\alpha \to 0$ it reduces to the standard algebra allowing bosonic mapping through the Holstein-Primakoff transformation (HPT): $S^+ = b^\dagger\sqrt{s - \rho\hat{N}}$, $S^- = \sqrt{s - \rho\hat{N}}b$, $S^3 = \hat{N} - \rho\frac{s}{2}$, $\hat{N} = b^\dagger b$, where $\rho = \pm$ correspond to $su(2)$ $(su(1,1))$. At this limit the xyz model reduces to the integrable elliptic Gaudin model [15] with mutually commuting conserved quantities

$$H_n = \sum_{a, m\neq n} w_a(nm) S_n^a S_m^a, \qquad (1)$$

with $w_1(\lambda) = \frac{cn}{sn}(\lambda)$, $w_2(\lambda) = \frac{dn}{sn}(\lambda)$, $w_3(\lambda) = \frac{1}{sn}(\lambda)$ and $\sum_n H_n = 0$, where we have introduced short hand notation $w_a(nm) \equiv w_a(z_n - z_m)$. However, we face now the difficulty, that for $\rho = +$ the operators $S^\pm$ in the HPT become nonhermitian at $< \hat{N} > > s$, while for $\rho = -$, though this is avoided, Gaudin Hamiltonian (1) becomes nonhermitian to maintain its integrability. We resolve this problem by taking the spin limit $s_0 \to \infty$ in HPT for the radiation field at $n = 0$, yielding

$$S_0^+ = \frac{1}{\epsilon}b^\dagger, \ S_0^- = \frac{1}{\epsilon}b, \ S_0^3 = -\frac{1}{2\epsilon^2}, \quad \epsilon = \frac{1}{\sqrt{s_0}} \to 0. \ (2)$$

We have considered here $\rho = +1$ for definiteness and kept terms up to order $O(\frac{1}{\epsilon})$. For modeling N_a two-level atoms we take spin $s_j = \frac{1}{2}$ representation at all other points $j = 1, 2, \ldots, N_a$. The $\frac{1}{\epsilon}$ coefficients in the expansion of the elliptic Gaudin Hamiltonians, though have the desired CRW terms, not yet yield the mutually commuting conserved set. The reason for this is twofold: the appearance of xyz type spin terms $\sigma_j^+ \sigma_k^+ + cc.$ with coefficients $w_-(jk) = \frac{1}{2}(w_1(jk) - w_2(jk))$ and the inhomogeneity $w_3(j0)$ in the σ_j^3 term, both of which we have to tackle before extracting integrable models from such an expansion. Obviously the first difficulty can be trivially resolved by setting $k = 0$, which yields $w_-(jk) = 0$ as considered in [16, 17]. However, since our aim is to keep the elliptic modulus k nontrivial, we have to take a different rout. Our strategy would be to push the undesired $w_-(jk)$ term out from the given order by suitably scaling the inhomogeneity parameters as $z_0 = K + \epsilon x_0$, $z_j = \epsilon x_j$, $j = 1, \ldots N_a$ and taking the limit $\epsilon \to 0$, K being the elliptic integral [19]. Observing that $su(2)$ is isomorphic under the reflection of any axis of the basis vectors, we also redefine the coupling constants in (1) as $w_2(jk) \to -w_2(jk)$, $w_1(jk) \leftrightarrow w_3(jk)$, and similarly for $w_a(0j)$, to have more conventional notation. The redefined coupling constants using $w_a(0j) = -w_a(j0)$ and the property of the elliptic functions, reduce in the needed order to

$$w_1(0j) \to 1, \ w_2(0j) \to k^{'}, \ w_3(0j) \to 0, \ w_a(jk) \to \frac{1}{\epsilon(x_j - x_k)}, \ a = 1, 2, 3, \ \text{with} \ k^{'} = (1 - k^2)^{\frac{1}{2}}. \qquad (3)$$

This shows that in the given order now we have $w_-(jk) = 0, w_3(j0) = 0$, which simultaneously removes both the above obstacles and derives finally from (1) in the first nontrivial order $O(\frac{1}{\epsilon})$, the integrable matter-radiation models with mutually commuting Hamiltonians $[H_j, H_k] = 0$:

$$H_j = \omega_a \sigma_j^3 + H_j^{b\sigma} + H_j^{\sigma\sigma}, \qquad (4)$$

with $j = 1, \ldots, N_a$, where

$$H_j^{b\sigma} = \Omega_+(b\sigma_j^+ + b^\dagger\sigma_j^-) + \Omega_-(b^\dagger\sigma_j^+ + b\sigma_j^-), \quad (5)$$

with $\Omega_\pm = 1 \pm k^{'}$, describes interaction between atoms and the radiation with explicit CRW terms and

$$H_j^{\sigma\sigma} = \sum_{k\neq j}^{N_a} \frac{1}{x_j - x_k}(\sigma_j^+ \sigma_k^- + \sigma_j^- \sigma_k^+ + \sigma_j^3 \sigma_k^3), \quad (6)$$

models interatomic interactions, but without having any xyz type spin term. This crucial fact permits us to include the atomic inversion term with arbitrary coefficient ω_a in (4), since σ_j^3 commutes with the whole Hamiltonian. The coupling constant Ω_- for the interaction with CRW terms clearly vanishes at $k = 0$, recovering the earlier results with RW approximation [16, 17]. Note that, we can construct a series of integrable Hamiltonians by different combinations of the commuting set H_j. For example, a generalization of the Tavis-Cummings model [8] without RW approximation can be constructed as

$$H_0 = \sum_j H_j = \sum_j^{N_a}[\omega_a \sigma_j^3 + \Omega_+(b\sigma_j^+ + b^\dagger\sigma_j^-) + \Omega_-(b^\dagger\sigma_j^+ + b\sigma_j^-)], \qquad (7)$$

without having explicit interactions between the atoms.

At $k = 1$, when $\Omega_+ = \Omega_- = 1$, (7) reduces further to

$$H_1 = \sum_j^{N_a} (\omega_a \sigma_j^3 + (b + b^\dagger)(\sigma_j^+ + \sigma_j^-)), \tag{8}$$

which appears in the trapped ion model interacting with the center of mass motion, after taking the customary Lamb-Dicke limit. Notice that (8) is similar to the model studied in [12], if a bosonic number operator term is added to it. However, while the model in [12] becomes integrable only at the thermodynamic and the mean field limit of the atoms, our model (8) achieves this without going to such limits. Moreover, contrary to the popular belief, that such a model with finite N_a atoms is exactly solvable only under RW approximation and otherwise chaotic [13], we show it to be Bethe ansatz solvable in general, at least for $\omega_f = 0$.

Since our model is derived from the elliptic Gaudin model through some limiting procedure, we can obtain its Bethe ansatz formulation also from the related result [15], by taking suitably the limits of the parameters involved. One important consequence of this inheritance is that, the excited Bethe eigenstates $|M>$, as in the elliptic Gaudin model, are no longer arbitrary, but constrained by the total spin value $M = \sum_n s_n$. For our model therefore, we must have: $M = \frac{1}{2} N_a + s_0 \to 1/\epsilon^2$ due to the limit $s_0 = 1/\epsilon^2 \to \infty$ for the bosonic mode. This macroscopic excitation with high photon number is in conformity with the results of [12], indicating that we must be in the super-radiant phase with no possibility of phase transition, since due to $\omega_f = 0$ the critical coupling here would be $\sqrt{\omega_f \omega_a} = 0$. Inspite of the fact that, the investigation of [12] is valid only for nontrivial ω_f and its approach is totally different from ours, some apparent similarity between these results in the integrable case, perhaps may be explained by the point that of the thermodynamic limit adopted in [12] is mimicked by the high spin limit for the bosonic mode, considered in our model.

For constructing exact eigenstates and eigenvalues, we introduce scaling also for the two sets of Bethe parameters: $w_b = \epsilon l_b, b = 1, \dots \frac{N_a}{2}$ and $w_\alpha^0 = \epsilon l_\alpha^0 + K, \alpha = 1, \dots, s_0$, similar to those for the inhomogeneity parameters defined above. The exact eigenvalue for our matter-radiation models (4) is then derived from the limiting values of the Bethe ansatz result of [15] as

$$E_j^{(crw)} = \theta'_{11}(0) \left(\omega_a + \sum_{b=1}^{\frac{N_a}{2}} \frac{1}{(x_j - l_b)} + \frac{1}{2} \sum_{k \neq j}^{N_a} \frac{1}{(x_j - x_k)} + \frac{\theta''_{10}}{\theta_{10}}(0)(2x_j - x_0 - S) \right), \tag{9}$$

where $S = \epsilon^2 \sum_\alpha^{s_0} l_\alpha^0$. Possible nontrivial contribution in the given order may be obtained for S as $S = l^0$, when $l_\alpha^0 = l^0$ are degenerate for all $\alpha = 1, \dots, s_0$, which we consider here for definiteness. From the Bethe equations for l_α^0 we find a solution for these degenerate parameters as $l^0 = x_0$, while for the rest of the Bethe parameters $l_b, \quad b = 1, \dots, \frac{N_a}{2}$, one needs to solve the equations

$$\frac{1}{2} \sum_{k=1}^{N_a} \frac{1}{(x_k - l_b)} = \omega_a + \sum_{c \neq b}^{\frac{N_a}{2}} \frac{1}{(l_c - l_b)}, \tag{10}$$

for a given set of arbitrary inhomogeneity parameters $x_j, j = 1, \dots, N_a$. For practical relevance we have applied this Bethe ansatz result to a $N_a = 2$ ion model with CRW terms described by (4) and shown its exact energy spectrum for $j = 1$ in Fig. 1. Similar result can be obtained easily for $j = 2$, by interchanging the parameters $x_1 \leftrightarrow x_2$.

For a closer contact with physical systems we may include a radiative excitation term $\omega_f \hat{N}$, perturbatively, over our Bethe ansatz solvable models (4,7,8). For smaller values of field frequency ω_f, when the RW approximation worsens, the perturbative treatment would become more accurate. One can generalize the formulation of standard perturbation theory for applying it to integrable models with exact eigenvalues $E_n(\vec{l})$ like (9) and the corresponding normalized Bethe eigenstates $\psi(\vec{l})$, with Bethe parameters $\vec{l} \equiv (l_1^0, \dots, l_{s_0}^0, l_1, \dots, l_{\frac{N_a}{2}})$. The result would however depend on the concrete model and might be difficult to extract in the explicit form. For considering the bosonic number operator term we are interested in, denoting $N(\vec{l}, \vec{m}) = <\psi(\vec{l})\hat{N}\psi(\vec{m})>$, we can derive the first order perturbative correction to the exact eigenvalues as $\omega_f N(\vec{l}, \vec{l})$ and to the corresponding eigenstates as $\omega_f \sum_{\{\vec{m}\}} \frac{N(\vec{m}, \vec{l})}{E_n(\vec{l}) - E_n(\vec{m})} \psi(\vec{m})$, where the sum is over the whole solution set of the Bethe parameters.

Thus we have constructed and exactly solved multi-

important field excitation term perturbatively, over the exactly solvable models.
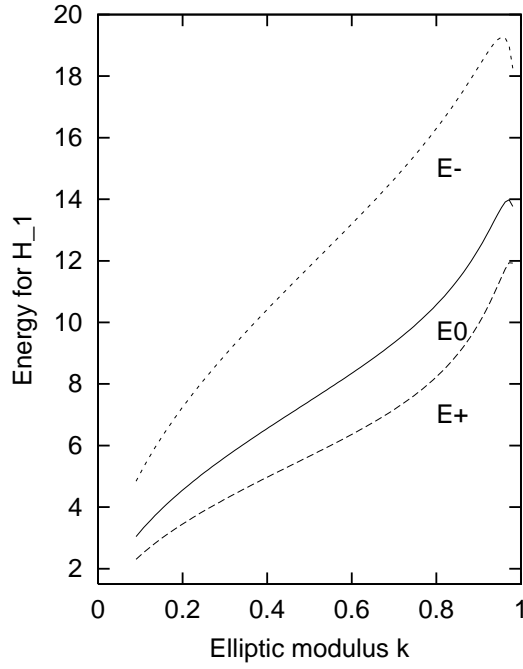

FIG. 1: Vacuum $E0$ and excitation energies $E\pm$ showing asymmetric Rabi-type energy splitting, for different values of elliptic modulus $k \in [0, 1]$. The graphs depends crucially on the inhomogeneity parameters (chosen here as $x_0 = 1, x_1 = 1.1, x_2 = 3$) and the atomic frequency (taken as $\omega_a = 1$). However the qualitative behavior appears to remain the same: all the energies rapidly increasing with the increase of k, with a curious dip toward the end.

atom matter-radiation models without rotating wave approximation and with explicit interatomic interactions. We derive our integrable models and the corresponding exact Bethe ansatz result from the elliptic Gaudin model through a limiting procedure. We can include physically

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